

①

Harmonic Potential in 1D

$$V = \frac{1}{2} k x^2$$

Time independent Schrödinger eq.

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} u(x) + \left(\frac{1}{2} k x^2 \right) u(x) = E u(x) \quad (1)$$

bound problem $\rightarrow$ a set of eigenstates

$$(1) \times -\frac{2m}{\hbar^2} \Rightarrow$$

$$\frac{d^2}{dx^2} u(x) + \frac{2mE}{\hbar^2} u(x) - \frac{mkx^2}{\hbar^2} u(x) = 0$$

Let $x = \alpha \xi \leftarrow$ dimensionless

$$\frac{d}{dx} = \frac{d\xi}{d\alpha d\xi} = \frac{1}{\alpha} \frac{d}{d\xi}$$

$$\frac{1}{\alpha^2} \frac{d^2}{d\xi^2} u(\xi) + \left(\frac{2mE}{\hbar^2} - \frac{\alpha^2 m k \xi^2}{\hbar^2} \right) u(\xi) = 0 \quad (2)$$

~~(1)~~ (2) $\times \alpha^2 \Rightarrow$

$$\frac{d^2}{d\xi^2} u(\xi) + \left(\frac{\alpha^2 2mE}{\hbar^2} - \frac{\alpha^4 m k \xi^2}{\hbar^2} \right) u(\xi) = 0$$

2

let $\alpha^4 = \frac{\hbar^2}{mk}$

$$\frac{d^2 u(\xi)}{d\xi^2} + \left(\frac{\hbar}{\sqrt{mk}} \frac{2mE}{\hbar^2} - \xi^2 \right) u(\xi) = 0 \quad (3)$$

let $\lambda = \frac{2E}{\hbar} \sqrt{\frac{m_0}{k}}$

$$\text{So } u'' + (\lambda - \xi^2) u = 0$$

dimensionless
equation

$$E = \frac{\hbar}{2} \sqrt{\frac{k}{m_0}} \lambda$$

$$x = \alpha \xi$$

dimensionless
distance

$$\alpha = \left(\frac{\hbar^2}{mk} \right)^{1/4}$$

$$u' = \frac{du(\xi)}{d\xi} \quad u'' = \frac{d^2 u(\xi)}{d\xi^2}$$

Look at (4)

and consider the case when $\xi \rightarrow \pm \infty$

To leading order $u_0 = e^{\pm \xi^2/2}$
satisfies this equation

(3)

Proof:

$$u_0' = \pm \xi e^{\pm \frac{1}{2}\xi^2} = \pm \xi u_0$$

$$u_0'' = \xi^2 u_0 \pm u_0$$

and

$$u_0'' + (1 - \xi^2) u_0$$

$$= (\cancel{\xi^2} \pm 1) u_0 + (1 - \cancel{\xi^2}) u_0$$

getting rid of, at least the ξ^2 term, but not the rest

Since the boundary condition $u(\xi) \rightarrow 0$
as $\xi \rightarrow \pm\infty$ ($x \rightarrow \pm\infty$)

we choose the - sign in the exponential

So, try a candidate sol'n of the form

$$u(\xi) = H(\xi) e^{-\frac{1}{2}\xi^2} \quad (5)$$

where $H(\xi)$ is not specified yet

(5) $\rightarrow$ (4) will give us a 2nd order differential eq'n for $H(\xi)$

(4)

$$\begin{aligned}
 u' &= H' e^{-\xi^2/2} - \xi H e^{-\xi^2/2} \\
 u'' &= H'' e^{-\xi^2/2} - \xi H' e^{-\xi^2/2} - \xi H' e^{-\xi^2/2} + \xi^2 H e^{-\xi^2/2} - H e^{-\xi^2/2} \\
 &= H'' e^{-\xi^2/2} - 2\xi H' e^{-\xi^2/2} + \xi^2 H e^{-\xi^2/2} - H e^{-\xi^2/2}
 \end{aligned}$$

(6)

(6) $\rightarrow$ (4) $\Rightarrow$

$$\left[(H'' - 2\xi H' + \xi^2 H - H) + (1 - \xi^2) H \right] e^{-\xi^2/2} = 0$$

or

$$\boxed{H'' - 2\xi H' + (1 - \xi^2) H = 0} \quad (7)$$

This is a well-studied differential equation. The solutions are called Hermite polynomials. Let's see how these are generated

Let's assume that H has a series form

$$H(\xi) = \xi^S (a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + \dots) \quad (8)$$

where $a_0 \neq 0$ and $S \geq 0$
to keep the solution
finite at $\xi = 0$

⑦ is valid $\forall s$, so, if we ~~⑧~~ $\rightarrow$ ⑦ we get relationships among the coefficients

Let's do that

$$H'' = (s(s-1)a_0 s^{s-2} + (s+1)s a_1 s^{s-1} + (s+2)(s+1)a_2 s^s + (s+3)(s+2)a_3 s^{s+1} \dots)$$

$$-2sH' = -2s(s a_0 s^{s-1} + (s+1)a_1 s^s + (s+2)a_2 s^{s+1} + (s+3)a_3 s^{s+2} \dots)$$

$$+ (1-s)H = (1-s)(a_0 s^s + a_1 s^{s+1} + a_2 s^{s+2} + a_3 s^{s+3} \dots) = 0$$

The coefficients of each power of s must vanish to be true for all s

Thus

- ⑨a $s(s-1)a_0 = 0$ coeff of s^{s-2}
- ⑨b $(s+1)s a_1 = 0$ " " s^{s-1}
- ⑨c $(s+2)(s+1)a_2 - (2s+1-1)a_0 = 0$ s^s
- ⑨d $(s+3)(s+2)a_3 - (2s+3-1)a_1 = 0$ s^{s+1}
- ⋮ etc

$$\textcircled{10} (s+n+2)(s+n+1)a_{n+2} - (2s+2n+1-1)a_n = 0$$

coefficient of s^{s+n}

where n is an integer

⑥

Proof of last step, for $\mathcal{S}^{SN}$

$$H'' = \frac{d^2}{dx^2} \left(\dots a_{N+2} \mathcal{S}^{SN+2} \dots \right)$$

$$-2\mathcal{S}H' = -2\mathcal{S} \frac{d}{dx} \left(\dots a_N \mathcal{S}^{SN} \right)$$

$$+ (1-\lambda)H = (1-\lambda) \left(\dots a_N \mathcal{S}^{SN} \right) = 0$$

$$\Rightarrow (SN+2)(SN+1)a_{N+2} - (2(SN)a_N + (1-\lambda)a_N) = 0$$

$$\text{or } (SN+2)(SN+1)a_{N+2} - (2S+2N+1-\lambda)a_N = 0$$

$\nwarrow$
 proof of ⑩

$$(9a) \quad s(s-1)a_0 = 0 \quad a_0 \neq 0, s \geq 0$$

$$\Rightarrow s=0 \text{ or } s=1 \text{ or both}$$

$$(9b) \quad s(s+1)a_1 = 0$$

$$\Rightarrow s=0 \text{ or } a_1=0 \text{ or both}$$

note $s \neq -1 \Rightarrow$ singular soln @ $x=0$

(10) connects a_{2+2} to a_2

So, if $s=0$, a_1 can be arbitrary. So,

$$\begin{aligned} H(\xi) &= a_0 \left[1 + \frac{a_2}{a_0} \xi^2 + \frac{a_4}{a_0} \xi^4 + \dots \right] \\ &\quad + a_1 \xi \left[1 + \frac{a_3}{a_1} \xi^2 + \frac{a_5}{a_1} \xi^4 + \dots \right] \\ &= H_e(\xi) + H_o(\xi) \end{aligned}$$

where $H_e(-\xi) = H_e(\xi)$ even soln
 $H_o(-\xi) = -H_o(\xi)$ odd soln

and, since $s=0$

$$(10) \Rightarrow (2n+2)(2n+1)a_{2n+2} = a_n(2n+1-1) \quad (11)$$

(9)

For the even solution, let $\nu = 2n$, $n \geq 1$.

(11) $\Rightarrow$

(12a)

$$(2n+2)(2n+1)a_{2n+2} = a_{2n}(4n+1-1)$$

$$n = 0, 1, 2, \dots$$

For the odd solution, let $\nu = 2n+1$, $n = 0, 1, 2, \dots$

(11) $\Rightarrow$

(12b)

$$(2n+3)(2n+2)a_{2n+3} = a_{2n+1}(4n+3-1)$$

For $S=1$, we must have $a_1 = 0$ from (9b) and have $a_3 = a_5 = a_7 = \dots = 0$ by (10)

$$\therefore H(\frac{1}{3}) = a_0' \left(1 + \frac{a_2'}{a_0'} \frac{1}{3^2} + \frac{a_4'}{a_0'} \frac{1}{3^4} + \dots \right) \text{ above}$$

— primes indicate different a 's from above

where $H(\frac{1}{3}) = -H(\frac{1}{3})$ an odd solution

Therefore from (10) $S=1 \Rightarrow$ let $\nu = 2n$

(12c)

$$(2n+3)(2n+2)a_{2n+2}' = a_{2n}'(4n+3-1)$$

But, this is exactly the odd part of the sol'n above. (identify $a_{2n+2}' = a_{2n+3}$)

(9)

So, this is not an independent sol'n. It is $H_0(z)$

So, apart from an overall multiplicative factor $H'(z) = H_0(z)$

We can combine these relations as follows.

$J = \text{even}$ even solution

$J = \text{odd}$ odd solution

To emphasize its integral form, let $J = m$

Going back to (11)

~~$$\frac{a_{m+2}}{a_m} = \frac{(m+2)(m+1)}{2m+1}$$~~

$$(13) \quad \frac{a_{m+2}}{a_m} = \frac{2m+1}{(m+2)(m+1)}$$

If $m \rightarrow \infty$

$$\frac{a_{m+2}}{a_m} \rightarrow \frac{2}{m}$$

But

$$e^{-z^2} = \sum_{\substack{m=0 \\ \text{even } m}}^{\infty} b_m z^m = \sum_{\substack{m=0 \\ \text{even } m}}^{\infty} \frac{z^m}{n!}$$

$$\therefore \frac{b_{m+2}}{b_m} = \frac{1}{m}$$

So, this means that unless the series in (13) terminates, we can not satisfy the boundary condition $\psi(\pm\infty) = 0$

$\therefore$ We have identified our eigenvalues λ_m

$$(14) \quad \lambda_m = 2m+1 \quad m=0,1,2,\dots$$

Or, going back to

$$(15) \quad \boxed{E_m = \hbar \sqrt{\frac{k}{m_0}} \left(m + \frac{1}{2}\right)} \quad m=0,1,2,\dots$$

The energy spacings are even!

And $\boxed{E_0 = \frac{1}{2} \hbar \sqrt{\frac{k}{m_0}}} \quad (15a) \quad \text{lowest ground state}$

(11)

A classical "spring" follows the equation

$$\ddot{x} + \frac{k}{m_0} x = 0$$

with solutions

$$x = A \cos \omega t + B \sin \omega t \quad (16)$$

$$\omega = \sqrt{\frac{k}{m_0}}$$

So, ω is like the classical oscillation frequency of an object of mass m_0 , with a spring constant k

$$F = m \ddot{x} = -kx$$

Let's solve this problem completely

$$\text{Apply the b.c. } x(0) = 0 \quad (17)$$

$$(17) \rightarrow (16) \Rightarrow A = 0$$

at (16) $\Rightarrow$

$$\ddot{x} = B\omega \cos \omega t$$

$$\ddot{x}(0) = B\omega$$

However at $x=0$, there are ~~not~~ forces on the particle, hence.

$$\frac{m}{2} \ddot{x}^2(0) = E = \frac{m}{2} B^2 \omega^2$$

$$B^2 = \frac{2E}{m\omega^2}$$

$$B = \frac{1}{\omega} \sqrt{\frac{2E}{m_0}}$$

$$\text{So } x = \frac{1}{\omega} \sqrt{\frac{2E}{m_0}} \sin \omega t$$

This identify $x_{\max}$ as the maximum distance from the origin at

$$\boxed{x_{\max} = \frac{1}{\omega} \sqrt{\frac{2E}{m_0}}} \quad (18)$$

We'll return to this later

(12)

Clearly $\langle x \rangle = 0$

What about $\langle x^2 \rangle$?

$$x = \underbrace{\frac{1}{\omega} \sqrt{\frac{2E}{m_0}}}_{\uparrow B} \sin \omega t$$

$$\langle x^2 \rangle = \frac{1}{\tau} \int_0^\tau dt x^2$$

$$\tau = 1 \text{ cycle} = \frac{2\pi}{\omega}$$

$$\begin{aligned} \text{So } \langle x^2 \rangle &= \frac{1}{\frac{2\pi}{\omega}} \int_0^{\frac{2\pi}{\omega}} dt B^2 \sin^2 \omega t \\ &= \frac{1}{2\pi} B^2 \underbrace{\int_0^{2\pi} d\theta \sin^2 \theta}_{=\pi} \\ &= \frac{1}{2} B^2 \end{aligned}$$

$$\langle x^2 \rangle = \frac{1}{2} \frac{1}{\omega^2} \sqrt{\frac{2E}{m_0}}^2 = \frac{E}{m_0 \omega^2}$$

(10)

$$\boxed{\langle x^2 \rangle = \frac{E}{m_0 \omega^2} = \frac{x_{\text{max}}^2}{2}}$$

Now, back to the wavefunctions.

We have, from (11)

$$Q_{v+2} = Q_v \frac{(2v+1 - \lambda_m)}{(v+2)(v+1)}$$

Q_0, Q_1 are arbitrary.

It is also clear that $H(\xi)$ is a polynomial of degree m , since $Q_{m+2} = 0$.

Convention

$$Q_0 = 1 \Rightarrow H_0(\xi) = 1$$

$$Q_1 = 2\xi \Rightarrow H_1(\xi) = 2\xi$$

and arrange for the leading power of the polynomial to have the form $(2\xi)^m$

$$\text{So, } \frac{Q_2}{Q_0} =$$

Let's generate $H_2(\xi)$

In this case $\lambda_2 = 5$

$$\frac{a_2}{a_0} = \frac{1-5}{2} = -2$$

$$\text{So, } H_2(\xi) = a_0 (1 - 2\xi^2)$$

By convention $a_0 = -2$ So

$$\boxed{H_2(\xi) = 4\xi^2 - 2}$$

Let's generate H_3 , $\lambda = 7$

$$\frac{a_3}{a_1} = \frac{3-7}{3 \cdot 2} = -\frac{4}{6} = -\frac{2}{3}$$

$$H_3(\xi) = a_1 \xi \left(1 + \frac{a_3}{a_1} \xi^2 \right)$$

$$= a_1 \xi \left(1 - \frac{2}{3} \xi^2 \right)$$

$$= a_1 \left(\xi - \frac{2}{3} \xi^3 \right)$$

apply convention

$$a_1 = \cancel{-12} -12$$

$$\boxed{H_3(\xi) = 8\xi^3 - 12\xi}$$

and
soon

α , one more

$$H_4(\xi)$$

$$\lambda = 9$$

$$\frac{a_2}{a_0} = \frac{1-9}{2} = -4$$

$$\frac{a_4}{a_0} = \left(\frac{a_4}{a_2} \right) \left(\frac{a_2}{a_0} \right)$$

$$= \left(\frac{5-9}{4 \cdot 3} \right) (-4)$$

$$= \frac{4}{3}$$

$$\text{So, } H_4(\xi) = a_0 \left[1 + \frac{a_2}{a_0} \xi^2 + \frac{a_4}{a_0} \xi^4 \right]$$

$$= a_0 \left[1 - 4\xi^2 + \frac{4}{3}\xi^4 \right]$$

By convention $a_0 = 12$

$$\boxed{H_4(\xi) = 16\xi^4 - 48\xi^2 + 12}$$

etc.

Obviously, this is a big pain!

There are ~~three more~~ techniques to do this, that are popular. I won't derive them, but I can give you notes on them.

(19)

$$H_m(z) = (-1)^m e^{z^2} \frac{d^m}{dz^m} e^{-z^2}$$

This allows us to repeatedly differentiate to get the H_m 's.

(20)

$$e^{z^2} = \sum_{m=0}^{\infty} H_m(z) \frac{z^m}{m!}$$

This means, if we expand the LHS of in a Taylor series, the coefficients will be the H_m 's $\div$ by $m!$.

The most useful form for computation involves the recursion relations

$$(21) \quad 0 = H_{m+1}(\xi) + 2m H_{m-1} - 2\xi H_m(\xi)$$

$$(22) \quad H_{m+1}(\xi) = 2\xi H_m(\xi) - H_m'(\xi)$$

Let's use (21) to generate the first few with the start

$$H_0(\xi) = 1$$

$$H_1(\xi) = 2\xi$$

Rewrite (21) as

$$(23) \quad H_m(\xi) = 2\xi H_{m-1}(\xi) - 2(m-1)H_{m-2}(\xi)$$

$$m=2 \quad H_2(\xi) = 2\xi H_1(\xi) - 2H_0(\xi) \\ = 4\xi^2 - 2 \quad \checkmark$$

$$m=3 \quad H_3(\xi) = 2\xi H_2(\xi) - 4H_1(\xi) \\ = (2\xi)(4\xi^2 - 2) - 4 \cdot 2\xi \\ = 8\xi^3 - 12\xi \quad \checkmark$$

$$\begin{aligned}
 m=4 \quad H_4(\xi) &= 2\xi H_3(\xi) - 2 \cdot 3 H_2(\xi) \\
 &= 2\xi(8\xi^3 - 12\xi) - 6(4\xi^2 - 2) \\
 &= 16\xi^4 - 48\xi^2 + 12 \quad \checkmark
 \end{aligned}$$

etc.

Here's a recursive function in C++

```

double hermite(double x, int m)
{
    if (m == 0) return 1;
    else if (m == 1) return 2 * x;
    else
        return 2 * x * hermite(x, m-1)
            - 2 * (m-1) * hermite(x, m-2);
}

```

We are not done yet ...

normalization

normalization

The Hermitees are normalized by their leading power, for mathematical convenience. We have to normalize the wave functions, so that we can compute expectation values.

So, we require

$$1 = N^2 \int_{-\infty}^{\infty} dx \psi^*(x) \psi(x)$$

$$= \frac{N^2}{\alpha} \int_{-\infty}^{\infty} d\xi \psi^*(\xi) \psi(\xi)$$

recall $\alpha = \left(\frac{\hbar^2}{m k} \right)^{1/4}$

$$\psi(\xi) = e^{-\xi^2/2} H_n(\xi)$$

(24)

$$\text{So, } \alpha = N^2 \int_{-\infty}^{\infty} d\xi e^{-\xi^2} H_n^2(\xi)$$

(20)

To normalize, go back to (20)

and write

$$F(\xi, s) = e^{s(2\xi - s)} = \sum_{m=0}^{\infty} H_m(\xi) \frac{s^m}{m!}$$

$$G(\xi, s') = e^{s'(2\xi - s')} = \sum_{m'=0}^{\infty} H_{m'}(\xi) \frac{s'^{m'}}{(m')!}$$

And

$$\int_{-\infty}^{\infty} d\xi e^{-\xi^2} F(\xi, s) G(\xi, s')$$

$$= \sum_{m, m'} \int_{-\infty}^{\infty} d\xi e^{-\xi^2} H_m(\xi) H_{m'}(\xi) \frac{s^m s'^{m'}}{m! (m')!}$$

$$= \int_{-\infty}^{\infty} d\xi e^{s(2\xi - s)} e^{s'(2\xi - s')} e^{-\xi^2}$$

$$= e^{2ss'} e^{-\frac{1}{4}(s-s')^2}$$

$$= e^{2ss'} \int_{-\infty}^{\infty} d\xi e^{-\frac{1}{4}(s-s-s')^2}$$

$$= \sqrt{\pi}$$

just
change
variables

$$\sum_{m, m'} \int_{-\infty}^{\infty} d^3z e^{-z^2} H_{m'}(z) H_m(z) \frac{s^m s'^{m'}}{m! (m')!}$$

$$= \sqrt{\pi} \left(\sum_m \frac{(ss')^m}{m!} \right)$$

All the coefficients in ss' must agree

$$\therefore \int_{-\infty}^{\infty} d^3z e^{-z^2} H_{m'}(z) H_m(z) = 0$$

for $m' \neq m$
(orthogonality)

$$\sum_m \int_{-\infty}^{\infty} d^3z e^{-z^2} H_m^2(z) \frac{(ss')^m}{(m!)^2}$$

$$= \sqrt{\pi} \sum_m \frac{(2ss')^m}{m!}$$

(25)

$\Rightarrow$

$$\int_{-\infty}^{\infty} d^3z e^{-z^2} H_m^2(z) = \sqrt{\pi} 2^m m!$$

$$(25) \rightarrow (24) \Rightarrow$$

$$\alpha = N^2 \sqrt{\pi} 2^m m!$$

$$Q \quad \boxed{N_m^2 = \frac{\alpha}{\sqrt{\pi} 2^m m!}}$$

Putting it all together

$$\psi_m(x) = \sqrt{\frac{\alpha}{2^m m! \sqrt{\pi}}} e^{-\frac{\alpha x^2}{2}} H_m(\alpha x)$$

$$\alpha = \left(\frac{\hbar^2}{m_0 k} \right)^{1/4} = \sqrt{\frac{\hbar}{m_0 \omega}}$$

$$E_m = \hbar \sqrt{\frac{k}{m_0}} \left(m + \frac{1}{2} \right)$$

$\omega = \sqrt{\frac{k}{m_0}} = \text{classical frequency} = \hbar \omega \left(m + \frac{1}{2} \right)$

$$H_0\left(\frac{x}{\xi}\right) = 1$$

$$H_1\left(\frac{x}{\xi}\right) = 2\xi$$

$$H_m\left(\frac{x}{\xi}\right) = 2\xi H_{m-1}\left(\frac{x}{\xi}\right) - 2(m-1) H_{m-2}\left(\frac{x}{\xi}\right)$$

Let's introduce a short hand

$$\langle H_{m'} | H_m \rangle$$

$$= \int_{-\infty}^{\infty} dx e^{-x^2} H_{m'}(x) H_m(x)$$

From (25) $\langle H_{m'} | H_m \rangle = \sqrt{\pi} 2^m m! \delta_{mm'}$

Kronecker delta

$$\delta_{mm'} = 0 \quad m \neq m'$$

$$= 1 \quad m = m'$$

Let's calculate

$\langle x^2 \rangle$ for the m th wavefunction

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \psi_m^*(x) x \psi_m(x)$$

$$= \frac{\alpha}{2^m m! \sqrt{\pi}} \int_{-\infty}^{\infty} d\zeta \zeta H_m^2(\zeta) e^{-\zeta^2}$$

$$\langle x \rangle = \frac{\alpha}{2^m m! \sqrt{\pi}} \langle H_m \times H_m \rangle$$

From (21)

$$\zeta H_m = \frac{1}{2} (H_{m+1} + 2m H_{m-1})$$

$$\therefore \langle H_m \zeta H_m \rangle = \frac{1}{2} \left\{ \langle H_m | H_{m+1} \rangle + 2m \langle H_m | H_{m-1} \rangle \right\}$$

by orthogonality

$$\therefore \langle x \rangle = 0!$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \psi_m^* x^2 \psi_m$$

$$= \frac{1}{2^m m! \sqrt{\pi}} \langle H_m x^2 H_m \rangle$$

From (2)

$$x^2 H_m = \frac{1}{4} \left\{ H_{m+2} + 4m^2 H_m + 4m H_{m-1} H_{m-1} \right\}$$

$$\langle H_m x^2 H_m \rangle = \frac{1}{4} \left\{ \sqrt{\pi} 2^{m+1} (m+1)! + 4m^2 \sqrt{\pi} 2^{m-1} (m-1)! \right\}$$

other term vanishes due to orthogonality

$$\text{So } \langle x^2 \rangle = \frac{1}{2^m m! \sqrt{\pi}} \frac{1}{4} \left\{ \sqrt{\pi} 2^{m+1} (m+1)! + 4m^2 \sqrt{\pi} 2^{m-1} (m-1)! \right\}$$

(26)

$$\begin{aligned}\langle x^2 \rangle &= x^2 \left(m \frac{1}{2} \right) \\ &= \frac{\hbar}{m\omega} \left(m \frac{1}{2} \right)\end{aligned}$$

$$\boxed{\langle x^2 \rangle = \frac{E_m}{m\omega^2}} \quad (26)$$

So

$$\langle x^2 \rangle_{\text{quantum}} = \langle x^2 \rangle_{\text{classical}}$$

$$E_m \rightleftharpoons E \quad !$$

Pretty remarkable result

Finally, what is α ?

$$\alpha = \sqrt{\frac{\hbar}{m\omega}} \text{ is a characteristic distance}$$